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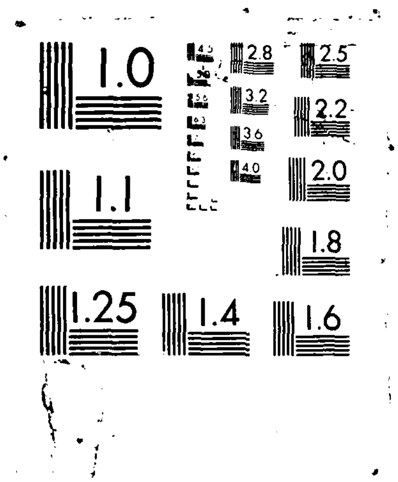
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Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

ON STABLE MARKOV PROCESSES

by

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157. G. Kallianpur and R. Wolpert, Weak convergence of stochastic neuronal models, Oct. 86. Stochastic Methods in Biology, M. Kimura et al., eds., Lecture Notes in Biomathematics, 70, Springer, 1987, 116-145.

158. G. Kallianpur, Stochastic differential equations in duals of nuclear spaces with some applications, Oct. 86. Inst. of Math. & Its Applications, 1986.

159. G. Kallianpur and R.L. Karandikar, The filtering problem for infinite dimensional stochastic processes, Jan. 87. Proc. Workshop on Stochastic Differential Systems, Stochastic Control Theory & Applications, Springer, to appear.

160. V. Perez-Abreu, Multiple stochastic integrals and nonlinear fractionals of a nuclear space valued Wiener process, Oct. 86. Appl. Math. Optimization, 16, 1987, 227-245.

161. R.L. Karandikar, On the Feynman-Kac formula and its applications to filtering theory, Oct. 86. Appl. Math. Optimization, to appear.

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Necessary conditions are given for a symmetric α-stable (SαS) process, $1<\alpha<2$, to be Markov. These conditions are then applied to find Markov or weakly Markov processes within certain important classes of SαS processes: time changed Levy motion, sub-Gaussian processes, moving averages and harmonizable processes. Two stationary SαS Markov processes are introduced, the right and the left SαS Ornstein-Uhlenbeck processes. Some of the results are in sharp contrast to the Gaussian case $\alpha=2$.

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Abstract: Necessary conditions are given for a symmetric α-stable (SαS) process, $1 < \alpha < 2$, to be Markov. These conditions are then applied to find Markov or weakly Markov processes within certain important classes of SαS processes: time changed Lévy motion, sub-Gaussian processes, moving averages and harmonizable processes. Two stationary SαS Markov processes are introduced, the right and the left SαS Ornstein-Uhlenbeck processes. Some of the results are in sharp contrast to the Gaussian case α=2.

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Key words and phrases: Markov and weakly Markov stable processes, time changed Lévy motion, sub-Gaussian and harmonizable processes, moving averages, left and right stable Ornstein-Uhlenbeck processes, stable conditional distributions.

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1. INTRODUCTION

Throughout $X = \{X(t),\ t \in T\}$ is a real symmetric α-stable (SαS) process with $0 < \alpha \leq 2$ and T an interval on the real line; i.e. all finite linear combinations $\Sigma_{n=1}^{N} a_n X(t_n)$ are SαS random variables.

X is Markov if for all $s \leq t$ in T, the conditional distribution of X(t) given X(u), $u \leq s$, "coincides" with the conditional distribution of X(t) given X(s) alone, in the sense that for any $u_i < s$, $i=1,\ldots,n$, and any Borel set E the equality $P\{X(t) \in E \mid X(u_1), \ldots, X(u_n), X(s)\} = P\{X(t) \in E \mid X(s)\}$ holds with probability 1. As is well-known the Markovian property is equivalent to the conditional independence of the past $\sigma\{X(u), u \leq t\}$ and future $\sigma\{X(u), u \geq t\}$ σ-fields given the present $\sigma\{X(t)\}$, and thus it is symmetric in time and could be defined by requiring that for all $t \leq s$ in T the conditional distribution of X(t) given X(u), $u \geq s$, "coincides" with the conditional distribution of X(t) given X(s) alone. Conditional distributions of non-Gaussian stable processes are generally very difficult to compute (and generally not stable) and it is thus not easy to check for the Markovian property. For this reason we introduce a weaker Markovian property which is amenable to some analysis and which concentrates on regressions.

For $1 < \alpha \leq 2$ we have $\mathscr{E}|X(t)| < \infty$ and we say that X is *left weakly Markov* if for all $s \leq t$ in T with probability 1,

$$\mathscr{E}\{X(t) \mid X(u), u \leq s\} = \mathscr{E}\{X(t) \mid X(s)\},$$

and *right weakly Markov* if for all $s \leq t$ with probability 1,

$$\mathscr{E}\{X(s) \mid X(u), u \geq t\} = \mathscr{E}\{X(s) \mid X(t)\}.$$

In the Gaussian case $\alpha=2$ the left and right weak Markovian properties are equivalent, and they are also equivalent to the Markovian property. Furthermore, there is only one stationary Gaussian process which is Markov, namely the Ornstein–Uhlenbeck process with covariance function $R(t) = R(0)e^{-\lambda|t|}$. In contrast there are stationary non–Gaussian $S\alpha S$ processes with $1 < \alpha < 2$ which:

(i) are left weakly Markov, without being right weakly Markov, and vice versa (cf. Section 7);

(ii) are left and right weakly Markov without being Markov, e.g. the sub–Ornstein–Uhlenbeck processes (Corollary 4.2);

(iii) are Markov, namely the $S\alpha S$ Ornstein–Uhlenbeck processes in (2.14) whose covariation function is the nonsymmetric double exponential function in (2.15);

(iv) have the symmetric covariation function $R(t) = R(0)e^{-\lambda|t|}$ but are neither left nor right weakly Markov, namely the harmonizable process in (6.3).

Two distinct $S\alpha S$ stationary Markov processes are identified in this paper. These are the right and the left $S\alpha S$ Ornstein–Uhlenbeck processes, which can be represented respectively as decreasing and increasing time changes of $S\alpha S$ Lévy motion (cf. (2.14) and (3.3)), or as nonanticipating and fully anticipating moving averages of $S\alpha S$ Lévy motion (Theorem 5.1 and (5.4)), and are the stationary solutions of certain first order stochastic differential equations driven by $S\alpha S$ white noise. Even though there might be further $S\alpha S$ stationary Markov processes, none is currently known. Such processes are not sub-Gaussian (Corollary 4.2) or harmonizable (Theorem 6.1); and they are neither nonanticipating nor fully anticipatory invertible moving averages, as the left and the right $S\alpha S$ Ornstein–Uhlenbeck processes are the only such $S\alpha S$ moving averages (Theorem 5.1 and page 33). Finally, neither one of their pairwise conditional distributions can be α–stable and symmetric, since the left and the right $S\alpha S$ Ornstein–Uhlenbeck processes are again the only ones possessing this property (Theorem 3.1). This may explain the difficulties in constructing other $S\alpha S$ stationary Markov processes, if indeed there are any.

Without requiring stationarity, the Gaussian case is still quite simple: All Gaussian Markov processes are essentially time changes of Brownian motion, see Tismoszyk (1974), Borisov (1982) and Wong and Hajek (1985). For non–Gaussian $S\alpha S$ processes with $1 < \alpha < 2$ the picture is more complex and rich. A necessary condition for left weak Markovianness is given in Theorem 2.1 in terms of the covariation function, and its solution is found, i.e. in the nonstationary case (2.5) and its generalization, and in the stationary case (2.9). While all time changes of Lévy motion have covariation function of this form (i.e. (2.5)), and are in fact Markov, they do not exhaust the class of $S\alpha S$ processes with covariation function of this form, e.g. Lévy bridge (see Example 2.1).

Time changes of Lévy motion are considered in Section 3 where they are shown to be the only $S\alpha S$ Markov processes whose pairwise conditional distributions are stable and symmetric (Theorem 3.1). In the non–Gaussian stable case there is also a marked asymmetry: The $S\alpha S$ Markov processes whose right to left and left to right pairwise conditional distributions are stable and symmetric are few and trivial when $1 < \alpha < 2$, whereas in the Gaussian case α=2 they coincide with the entire class of Gaussian Markov processes.

An auxiliary result of independent interest is given in Proposition 3.1 and Corollary 3.1, characterizing the stability of the conditional distribution(s) of random variables that are jointly $S\alpha S$.

Sub–Gaussian processes are left (right) weakly Markov if and only if they are essentially time changes of sub–Brownian motion, except for trivial cases, and they are not Markov (Theorem 4.2). In particular, the only weakly Markov stationary $S\alpha S$ sub–Gaussian processes are the sub–Ornstein– Uhlenbeck processes (Corollary 4.2).

Sections 5 and 6 consider two specific classes of stationary $S\alpha S$ processes, moving averages and harmonizable $S\alpha S$ processes. It is shown that in the case of either nonanticipating or fully anticipatory *invertible moving* averages, the weak Markov property cannot exist without full Markovianness and it is realized only by the right and the left $S\alpha S$ Ornstein–Uhlenbeck processes correspondingly (Theorem 5.1 and page 33). In sharp contrast to the Gaussian case α=2, it turns out that for the stable case

$1 < \alpha < 2$, weak Markovianness never prevails for harmonizable processes (Theorem 6.1).

Finally, a family of one-sided weakly Markov (i.e. left weakly Markov but not right weakly Markov or vice versa) SαS processes is constructed in Section 7.

2. GENERALITIES

A SαS process X can be represented by an integral of the form

$$X(t) = \int_U f(t,u)\, dZ(u), \quad t \in T, \tag{2.1}$$

where Z is a SαS random measure on some σ–finite measure space (U,Σ,μ) (i.e. Z is an independently scattered σ–additive set function on $\Sigma_\mu = \{E \in \Sigma,\ \mu(E) < \infty\}$ and $\mathcal{E}\exp\{irZ(E)\} = \exp\{-\mu(E)|r|^\alpha\}$ for $E \in \Sigma_\mu$) and $\{f(t,\cdot),\ t \in T\} \subset L_\alpha(U,\Sigma,\mu)$ (Kanter (1972), Kuelbs (1973) and Hardin (1982)). All quantities are real–valued (except in Section 6, where complex–valued processes are discussed) and μ is called the control measure of Z. For every $g \in L_\alpha(\mu)$ the integral $\int gdZ$ is a SαS r.v. with $\mathcal{E}\exp\{ir\int gdZ\} = \exp\{-|r|^\alpha \int |g|^\alpha d\mu\}$ and is linear in g. Specific examples of such representations of SαS processes will be considered in Sections 5 and 6.

The covariation function of X is

$$R(t,s) = \mathrm{Cov}[X(t), X(s)] = \int_U f(t,u)\, f(s,u)^{<\alpha-1>} d\mu(u), \tag{2.2}$$

where $x^{<q>} = |x|^q \mathrm{sgn}(x)$, and it does not depend on the specific representation of X (for more on the covariation see Section 3). In the Gaussian case, α=2, the covariation is a multiple of the covariance, R(t,t) determines the distribution of X(t), and the numbers R(t,t), R(s,s), R(t,s) determine the joint distribution of X(t), X(s); thus knowledge of R on T×T determines the distribution of the (zero–mean) Gaussian process X. In the non–Gaussian SαS case $1 < \alpha < 2$, the covariation is not generally a symmetric function of its arguments and is linear only in the first argument, R(t,t) determines the distribution of X(t), but the numbers R(t,t), R(s,s), R(t,s), R(s,t) do not generally determine the joint distribution of X(t), X(s). Thus knowledge of the covariation function R on T×T generally does not determine the bivariate distributions of the SαS process X. Still, as we shall see, the covariation function plays a role partially analogous to the role played by the covariance function in the Gaussian case.

A basic result on the left weak Markovian property is the following:

Theorem 2.1. *X is left weakly Markov if and only if*

$$\mathrm{Cov}[X(t) - \frac{R(t,s)}{R(s,s)} X(s), Y] = 0 \tag{2.3}$$

for all $s \le t$ *and all* $Y \in \overline{\mathrm{sp}}\{X(u), u \le s\}$, *where the closure is in probability. If* X *is left weakly Markov then*

$$R(t_3,t_2)\, R(t_2,t_1) = R(t_3,t_1)\, R(t_2,t_2) \qquad \text{for all} \quad t_1 \le t_2 \le t_3. \tag{2.4}$$

A covariation function R *with* $R(t,s) \neq 0$ *for all* $s < t$ *in* T *satisfies* (2.4) *if and only if it is of the form*

$$R(t,s) = H(t)\, K(s)^{<\alpha-1>} \qquad \text{for all} \quad s \le t, \tag{2.5}$$

where the functions K, H *are unique up to a multiplicative constant, have the same sign and* $K(t)/H(t)$ *is positive and nondecreasing on* T.

Proof. It is known (see Kanter (1972)) that

$$\mathcal{E}\{X(t)\,|\,X(s)\} = \frac{R(t,s)}{R(s,s)} X(s).$$

Therefore X is left weakly Markov iff

$$\mathcal{E}\{X(t)\,|\,X(u), u \le s\} = \frac{R(t,s)}{R(s,s)} X(s), \qquad \forall\ s < t,$$

and by [3, Proposition 1.5], a necessary and sufficient condition for this is (2.3) for all $s < t$ and $Y \in \overline{\mathrm{sp}}\{X(u), u \le s\}$.

Now if X is left weakly Markov, taking $Y = X(u)$, $u < s$, we obtain

$$R(t,u) = \frac{R(t,s)}{R(s,s)} R(s,u), \qquad \forall\ u < s < t,$$

which is (2.4). The general form (2.5) of the solution of (2.4) is obtained as in Borisov (1982) by taking, for some interior point t_0 of T, $K(t)^{<\alpha-1>} = R(t_0,t)$ for $t \le t_0$, $= R(t,t)\ R(t_0,t_0)/R(t,t_0)$ for $t > t_0$, and $H(t) = R(t,t)/R(t_0,t)$ for $t \le t_0$, $= R(t,t_0)/R(t_0,t_0)$ for $t > t_0$. Since by (2.2), $R(t,t) \ge 0$ and by assumption $R(t,t) \ne 0$, it follows from $0 < R(t,t) = H(t)K(t)^{<\alpha-1>}$ that K and H have the same sign at each point. Also from (2.2) and Hölder's inequality we obtain

$$|R(t,s)| \le \{R(t,t)\}^{1/\alpha} \{R(s,s)\}^{1-1/\alpha} \tag{2.6}$$

and substituting from (2.5) we have $|K(s)/H(s)| \le |K(t)/H(t)|$. Since KH^{-1} is positive, it is nondecreasing on T. Conversely, (2.5) implies (2.4) immediately and the property KH^{-1}: nondecreasing, is needed to show that R given by (2.5) is covariation function. The simplest way of showing this is by constructing a SαS process with covariation (2.5), as was done in the Gaussian case in Wong and Hajek (1985), p. 64. Indeed, using the time change $\tau(t) = \{K(t)H^{-1}(t)\}^{\alpha-1}$ (nondecreasing), and the SαS Lévy motion $L = \{L(t),\ t \ge 0\}$ which has stationary independent increments, $L(0) = 0$, and $\mathscr{E}\exp\{ir[L(t)-L(s)]\} = \exp\{-|r|^{\alpha}|t-s|\}$, we can introduce the S$\alpha$S process

$$X(t) = H(t)\ L(\tau(t)) \tag{2.7}$$

whose covariation function is for $s < t$,

$$\begin{aligned} \mathrm{Cov}[X(t),X(s)] &= H(t)H(s)^{<\alpha-1>}\ \mathrm{Cov}[L(\tau(t)),\ L(\tau(s))] \\ &= H(t)H(s)^{<\alpha-1>}\ \tau(s) = H(t)H(s)^{<\alpha-1>}\ \{\tfrac{K(s)}{H(s)}\}^{\alpha-1} \\ &= H(t)K(s)^{<\alpha-1>} = R(t,s) \end{aligned} \tag{2.8}$$

since $K(t)H(t) > 0$. □

In the Gaussian case $\alpha=2$, the covariation is linear in its second argument (as well as in its first), and the necessary condition (2.4) is also sufficient; thus when $R(t,t) \neq 0$, $t \in T$, conditions (2.3), (2.4), (2.5) and (2.7) are all equivalent, and all Gaussian Markov processes are time changes of Brownian motion. However, in the non–Gaussian SαS case with $1 < \alpha < 2$, generally the covariation is not linear in its second argument and the necessary condition (2.4) is not sufficient. Also, while the time changes of SαS Lévy motion (2.7) have covariation function of the form (2.5), they do not exhaust the class of SαS processes with covariation function of the form (2.5).

<u>Example 2.1</u> *Lévy bridge.*

Again let L be the Lévy motion, and let $B(t) = L(t) - tL(1)$, $0 \leq t \leq 1$. This is one of the possible generalizations of the Brownian bridge to the SαS case, $\alpha<2$. It is starightforward to check that for this process

$$R(t,s) = \mathrm{Cov}[B(t), B(s)] = \begin{cases} (1-t)s[(1-s)^{\alpha-1} + s^{\alpha-1}] & \text{if } 0 \leq s \leq t \leq 1, \\ t(1-s)[(1-s)^{\alpha-1} + s^{\alpha-1}] & \text{if } 0 \leq t \leq s \leq 1. \end{cases}$$

Moreover, B(t) is easily seen to satisfy the condition (2.3) for any $Y = \Sigma_{i=1}^{k} a_i X(u_i)$, $u_i \leq s$, for $i=1,2,\ldots,k$, and, therefore, for any $Y \in \overline{\mathrm{sp}}\{X(u),\ u \leq s\}$. This process is, therefore, left weakly Markov and, in fact, two–sided weakly Markov, since its right weak Markovianness can be established similarly.

The Lévy bridge B(t) is an example of a two–sided weakly Markov SαS process which is not a time changed Lévy motion (see (2.12)). Other examples of such processes are furnished by the sub–Gaussian SαS processes (see Section 4).

The process B(t) is probably not Markov. It is interesting to note that another possible generalization of the Brownian bridge, namely $B'(t) = (1-t)L[t/(1-t)]$, $0 \leq t \leq 1$, is clearly distinct from B(t) when $\alpha < 2$! B' is a Markov process, and its covariation function is given by

$$R'(t,s) = \mathrm{Cov}[B'(t), B'(s)] = \begin{cases} (1-t)s(1-s)^{\alpha-2} & \text{if } 0 \le s \le t \le 1, \\ t(1-s)^{\alpha-1} & \text{if } 0 \le t \le s \le 1. \end{cases}$$

Not much seems to be known about the role of these processes (if any) in the weak convergence of empirical processes. □

The solution of (2.4) in the general case, i.e. without the condition $R(t,t) \neq 0$ on T, can be obtained as in the Gaussian case (Timoszyk (1974), Borisov (1982)), in the form (2.5) on a finite or denumerable union of disjoint squares around the diagonal of $T \times T$ (and zero elsewhere).

When X is stationary ($T = \mathbb{R}^1$) then $R(t,s)$ depends only on $t-s$ and we write $R(t,s) = R(t-s)$. When $\alpha=2$ the converse is also true, but this is not generally true when $1 < \alpha < 2$. When $R(t,s) = R(t-s)$ for all $t,s \in \mathbb{R}^1$, we say that X is *covariation stationary.* In the presence of stationarity Theorem 2.1 reduces to the following simpler form.

Corollary 2.1. *Let* $T = \mathbb{R}^1$. *If* X *is covariation stationary and left weakly Markov, then for some* $0 \le \lambda \le \infty$,

(2.9) $$R(t) = R(0)e^{-\lambda t} \quad \textit{for all } t > 0.$$

If X *is stationary, then it is left weakly Markov if and only if*

(2.10) $$\mathrm{Cov}[X(t) - e^{-\lambda t}X(0), Y] = 0$$

for some $0 \le \lambda \le \infty$ *and all* $t > 0$, $Y \in \overline{\mathrm{sp}}\{X(u), u \le 0\}$.

Proof. If X is covariation stationary and left weakly Markov, then (2.4) is satisfied and can be written in the form

$$R(u)R(v) = R(u+v)R(0) \quad \text{for all} \quad u,v \ge 0.$$

Since by (2.6), $|R(t)| \le R(0)$, $\forall\, t$, $R(t)/R(0)$ is bounded and therefore the solutions of the above equation are given by (2.9) (see Feller (1968), p. 459) □

When $\lambda = 0$ in (2.9), $R(t) = R(0)$ for all $t > 0$, i.e. equality holds in Hölder's inequality (2.6), and thus for each pair $s < t$ we have $X(t) = X(s)$ a.s. Hence X is equal in law to a constant process $\{C(t) = aZ, -\infty < t < \infty\}$, $a > 0$, Z a standard SαS r.v., and every separable modification of X has constant paths. At the other extreme, when $\lambda = +\infty$, we have $R(t) = 0$ for $t > 0$ and $R(0) > 0$, so that the stationary process X is not continuous in probability and thus its sample functions do not have measurable modifications ([2], p. 3), i.e. it is very irregular. The interesting case then is when $0 < \lambda < \infty$. In the Gaussian case α=2, the symmetry of R and the fact that it determines the distribution of X, imply that the only stationary, Gaussian, left weakly Markov processes are the Ornstein–Uhlenbeck processes with covariance function $R(t) = R(0)e^{-\lambda|t|}$, $-\infty < t < \infty$, which are in fact Markov. As we shall see in the non–Gaussian SαS case $1 < \alpha < 2$ there exist left weakly Markov stationary processes that are not Markov (see e.g. Section 4).

Results analogous to Theorem 2.1 and Corollary 2.1 are clearly valid for the right weak Markovian property. We will not repeat the details here; we only mention that (2.4) takes the form

$$R(t_1,t_2)R(t_2,t_3) = R(t_1,t_3)R(t_2,t_2) \quad \text{for all } t_1 \le t_2 \le t_3,$$

and (2.5) takes the following form:

$$R(t,s) = H(s)K(t)^{<\alpha-1>} \qquad \text{for all } t \le s,$$

where $K(t)/H(t)$ is positive and nondecreasing on T.

Therefore, if X is two–sided weakly Markov with $R(t,t) \ne 0$ on T, then

$$(2.11) \qquad R(t,s) = \begin{cases} H_1(t)K_1(s)^{<\alpha-1>}, & s \le t, \\ H_2(t)K_2(s)^{<\alpha-1>}, & t \le s. \end{cases}$$

When $1 < \alpha < 2$ the two pairs of functions K_1, H_1, and K_2, H_2 need not be identical, as is the case with the time changes of SαS Lévy motion defined by (2.7) where

$$(2.12) \qquad H_2(t) = H_1(t)^{<2-\alpha>}K_1(t)^{<\alpha-1>}, \quad K_2(t) = H_1(t).$$

These Lévy motion time changes are in fact Markov, as follows from

$$X(t) = \frac{H(t)}{H(s)} X(s) + H(t)\ \{L(\tau(t)) - L(\tau(s))\},$$

and they have α-stable conditional distributions symmetric about $\{H(t)/H(s)\}X(s)$ for $s < t$:

$$\begin{aligned} \mathcal{E}\{\exp[irX(t)]\,|\,X(u),u\le s\} &= \exp\{ir\,\frac{H(t)}{H(s)}\,X(s)\}\,\mathcal{E}\exp\{irH(t)[L(\tau(t))-L(\tau(s))]\} \\ (2.13) \qquad &= \exp\{ir\,\frac{H(t)}{H(s)}\,X(s) - |r|^{\alpha}|H(t)|^{\alpha}[\tau(t)-\tau(s)]\} \\ &= \mathcal{E}\{\exp[irX(t)]\,|\,X(s)\}. \end{aligned}$$

In particular every two-sided weakly Markov stationary SαS process has

$$R(t) = \begin{cases} R(0)e^{-\lambda_1 t}, & t \ge 0, \\ R(0)e^{\lambda_2 t}, & t \le 0, \end{cases}$$

for some $0 \le \lambda_1,\lambda_2 \le \infty$. When $1 < \alpha < 2$, the exponents λ_1 and λ_2 need not be equal, see (2.15).

Among the time changes (2.7) of SαS Lévy motion the only stationary ones are of the form

$$(2.14) \qquad X(t) = a\,e^{-\lambda t}\,L(e^{\alpha\lambda t}), \quad -\infty < t < \infty,$$

for some $0 < a < \infty$, $0 \le \lambda < \infty$; and in fact it can be easily seen they are the only ones with stationary bivariate distributions (i.e. for these time changes, bivariate stationarity implies stationarity). When $0 < \lambda < \infty$ the Markov processes (2.14) will be called SαS *Ornstein–Uhlenbeck* with parameters α and λ. Using (2.7), (2.11) and (2.12) we conclude that the covariation function of the SαS Ornstein–Uhlenbeck process (2.14) is given by

$$R(t) = \begin{cases} R(0)e^{-\lambda t}, & t \geq 0, \\ R(0)e^{(\alpha-1)\lambda t}, & t \leq 0, \end{cases} \tag{2.15}$$

and is not symmetric unless α=2.

3. MORE ON TIME CHANGED LÉVY MOTION

In Section 2 we saw that time changes of SαS Lévy motion are Markov with conditional distribution of X(t) given X(s) α-stable and symmetric, for any $s < t$. Here we show that all SαS Markov processes with right to left conditional distributions (as above) α-stable and symmetric are made up from independent segments of time changed Lévy motion; and in particular the only stationary ones with dependent values are SαS Ornstein–Uhlenbeck processes.

Recall that a SαS Lévy motion $L = \{L(t),\ t \geq 0\}$ is a process with stationary independent increments, $L(0) = 0$ a.s., and for all real r and $t, s \geq 0$,

$$\mathscr{E} \exp\{ir[L(t)-L(s)]\} = \exp\{-|r|^{\alpha} |t-s|\}. \tag{3.1}$$

If $\tau(t)$ is positive and nondecreasing on T, and H(t) is positive on T, then the time change of the SαS Lévy motion

$$X(t) = H(t)\, L(\tau(t)), \qquad t \in T, \tag{3.2}$$

is Markov and for $s < t$, the conditional distribution of X(t) given X(s) is α-stable and symmetric, cf. (2.13).

Note that we can regard the above time change as *increasing* (the new clock $\tau(t)$ is an increasing function on T). Similarly, we can define a *decreasing* time change of SαS Lévy motion by taking the clock $\tau(t)$ in (3.2) to be a decreasing nonnegative function on T. Of course, the new class of SαS processes obtained in this way consists of Markov processes. Moreover, they have the following common property for $s > t$, the conditional distribution of X(t) given X(s) is α-stable and symmetric. These properties of the time changes of SαS Lévy motion are quite remarkable. We will see in this section that there are not many Markov SαS processes whose conditional distributions are α-stable and symmetric.

The conditional distributions of every Gaussian process are Gaussian and symmetric (around the conditional mean). Non–Gaussian stable processes in general do not have stable conditional distributions. Our aim in this section is to characterize the classes $\mathscr{M}_\alpha^{(\ell)}$ and $\mathscr{M}_\alpha^{(r)}$ of all SαS Markov processes X which have the property that for all $s < t$ ($s > t$, correspondingly) the conditional distribution of X(t) given X(s) is α–stable and symmetric. Of course if $1 < \alpha < 2$ and $\mathscr{L}\{X(t)|X(s)\}$ is symmetric about some point, this point of symmetry is necessarily the conditional mean $\mathscr{E}\{X(t)|X(s)\}$.

Theorem 3.1 characterizes the processes in $\mathscr{M}_\alpha^{(\ell)}$ and $\mathscr{M}_\alpha^{(r)}$ when $1 < \alpha < 2$ Those with covariation function nonvanishing everywhere are time changes of Lévy motion, as in (3.2). The general process in $\mathscr{M}_\alpha^{(\ell)}$ ($\mathscr{M}_\alpha^{(r)}$) is then made up of independent segments of Lévy motion time changes on disjoint intervals. There are two extreme (and uninteresting) cases: the (very smooth) constant process ($X(t) = Z$ a.s. for each t), and the (very rough) process consisting of independent random variables. In the stationary case the latter process would have independent and identically distributed SαS r.v.'s, with scale parameter $a > 0$, and we denote it by $I_a = \{I_a(t), -\infty < t < \infty\}$. The only stationary processes in $\mathscr{M}_\alpha^{(\ell)}$ are the SαS Ornstein–Uhlenbeck processes (2.14) and those with iid values: I_a. The only stationary processes in $\mathscr{M}_\alpha^{(r)}$ are the inverted SαS Ornstein–Uhlenbeck processes defined by

$$X(t) = ae^{\lambda t} L(e^{-\alpha\lambda t}), \quad -\infty < t < \infty, \ a > 0, \ \lambda \geq 0, \tag{3.3}$$

and the processes I_a. The SαS Ornstein–Uhlenbeck processes (2.14) and (3.3) coincide trivially in the Gaussian case $\alpha=2$, but not in the case $\alpha < 2$ (more on this point is said in Theorem 3.2).

In the statement of Theorem 3.1 equality in law, $\mathscr{L}$, means equality of all finite dimensional distributions.

Theorem 3.1. *Let X belong to* $\mathcal{M}_\alpha^{(\ell)}$ *(correspondingly,* $\mathcal{M}_\alpha^{(r)}$*) for some* $1 < \alpha < 2$.

(*i*) *If its covariation function satisfies* $R(t,s) \neq 0$ *for all* $s < t$ *in* T, *then*

$$\{X(t),\, t \in T\} \overset{\mathcal{L}}{=} \{H(t)L(\tau(t)),\, t \in T\}$$

for some positive, nondecreasing (correspondingly, nonincreasing) function τ *on* T, *and some positive function* H *on* T.

(*ii*) *If* X *is stationary then either*

$$\{X(t),\, -\infty < t < \infty\} \overset{\mathcal{L}}{=} \{ae^{-\lambda t}L(e^{\alpha\lambda t}),\, -\infty < t < \infty\}$$

for some $a > 0$ *and* $0 \leq \lambda < \infty$ *(correspondingly,* $0 \leq -\lambda < \infty$*), or else for some* $a > 0$,

$$\{X(t),\, -\infty < t < \infty\} \overset{\mathcal{L}}{=} \{I_a(t),\, -\infty < t < \infty\}.$$

In order to prove Theorem 3.1 we need the following properties of bivariate SαS distributions which are of independent interest. Let us recall that the r.v.'s X_1 and X_2 are jointly SαS if their joint characteristics function is of the form

$$\mathcal{E}\exp\{i(r_1X_1+r_2X_2)\} = \exp\{\, -\int_{S_2}|r_1x_1+r_2x_2|^\alpha d\Gamma(x_1,x_2)\, \}$$

for all real r_1,r_2, where Γ is a uniquely determined symmetric finite measure on the unit circle S_2 in $\mathbb{R}^2$. When $1 < \alpha < 2$, the covariation of X_1 with X_2 is given by

$$\mathrm{Cov}[X_1,\, X_2] = \int_{S_2} x_1x_2^{<\alpha-1>}d\Gamma(x_1,x_2)$$

[5] (which is consistent with (2.2)). We denote by $\|X_i\|_\alpha$ their scale parameter $\|X_i\|_\alpha^\alpha = \int_{S_2}|x_i|^\alpha d\Gamma(x_1,x_2) = \mathrm{Cov}[X_i,\, X_i]$, i=1,2, and we have by Kanter (1972),

$$\mathcal{E}(X_2|X_1) = \frac{\mathrm{Cov}[X_2, X_1]}{\mathrm{Cov}[X_1, X_1]} X_1 \overset{\Delta}{=} \rho_{21} X_1.$$

Proposition 3.1. *Let* X_1 *and* X_2 *be jointly* SαS *and* $1 < \alpha < 2$. *Then the following are equivalent.*

(*i*) $\mathcal{L}(X_2|X_1)$ is *α–stable and symmetric.*

(*ii*) $X_2 - \rho_{21} X_1$ is *independent of* X_1.

(*iii*) Γ *is concentrated on* $\pm(0,1)$, $\pm((1+\rho_{21}^2)^{-1/2}, \rho_{21}(1+\rho_{21}^2)^{-1/2})$.

Under any of these conditions we have

$$(3.4) \qquad \|X_2-\rho_{21}X_1\|_\alpha^\alpha = \|X_2\|_\alpha^\alpha - |\rho_{21}|^\alpha \|X_1\|_\alpha^\alpha = \|X_2\|_\alpha^\alpha - \frac{|\mathrm{Cov}[X_2, X_1]|^\alpha}{\|X_1\|^{\alpha(\alpha-1)}},$$

$$X_1, X_2 \textit{ are independent} \quad \textit{iff} \quad \mathrm{Cov}[X_2, X_1] = 0 \quad \textit{iff} \quad \rho_{21} = 0.$$

Proof. Assume (i). Then

$$\mathcal{E}\{\exp(ir_2X_2)|X_1\} = \exp\{-|r_2|^\alpha M(X_1) + ir_2 N(X_1)\},$$

for some real measurable functions M and N with $M \geq 0$. It follows that $N(X_1) = \mathcal{E}(X_2|X_1) = \rho_{21} X_1$. Let

$$Z = \rho_{21} X_1 + M(X_1)^{1/\alpha} Z_0$$

where Z_0 is independent of X_1 and $\mathcal{E}\exp(irZ_0) = \exp(-|r|^\alpha)$, $r \in \mathbb{R}^1$. Then clearly $\mathcal{L}(Z|X_1) = \mathcal{L}(X_2|X_1)$ and thus $\mathcal{L}(X_1,Z) = \mathcal{L}(X_1,X_2)$. It follows that $Z - \rho_{21}X_1$ is SαS, and thus for some $c \geq 0$ and every real r,

$$\exp(-|r|^\alpha c) = \mathcal{E}\exp\{ir(Z-\rho_{21}X_1)\} = \mathcal{E}\exp\{irM(X_1)^{1/\alpha}Z_0\} = \mathcal{E}\exp\{-|r|^\alpha M(X_1)\}.$$

By the uniqueness of the Laplace transform we conclude that $M(X_1) = c$ a.s.. Then

$Z - \rho_{21}X_1 = cZ_0$ is independent of X_1, and this implies that $X_2 - \rho_{21}X_1$ is also independent of X_1, proving (ii).

Conversely, assuming (ii) and writing $X_2 = (X_2 - \rho_{21}X_1) + \rho_{21}X_1$ we obtain, since $X_2 - \rho_{21}X_1$ is SαS,

$$\mathcal{E}\{\exp(ir_2X_2)\,|\,X_1\} = \exp\{-|r_2|^{\alpha}\|X_2-\rho_{21}X_1\|_{\alpha}^{\alpha} + ir_2\rho_{21}X_1\} \tag{3.5}$$

so that (i) is satisfied, and in fact the constant $M(X_1) = c$ is equal to $\|X_2-\rho_{21}X_1\|_{\alpha}^{\alpha}$. We also obtain

$$\|X_2\|_{\alpha}^{\alpha} = \|X_2-\rho_{21}X_1\|_{\alpha}^{\alpha} + |\rho_{21}|^{\alpha}\,\|X_1\|_{\alpha}^{\alpha},$$

from which (3.4) follows.

Therefore (i) or (ii) imply

$$\begin{aligned}\mathcal{E}\exp\{i(r_1X_1+r_2X_2)\} &= \mathcal{E}\exp\{i(r_1+r_2\rho_{21})X_1 - |r_2|^{\alpha}\|X_2-\rho_{21}X_1\|_{\alpha}^{\alpha}\} \\ &= \exp\{-|r_1+r_2\rho_{21}|^{\alpha}\|X_1\|_{\alpha}^{\alpha} - |r_2|^{\alpha}\|X_2-\rho_{21}X_1\|_{\alpha}^{\alpha}\}\end{aligned}$$

from which (iii) is evident. And conversely, assuming (iii) we have

$$\mathcal{E}\exp\{i(r_1X_1+r_2X_2)\} = \exp\{-|r_2|^{\alpha}d_1 - |r_1+r_2\rho_{21}|^{\alpha}d_2\}$$

for some $d_1, d_2 \geq 0$, and therefore

$$\begin{aligned}\mathcal{E}\{i[a(X_2-\rho_{21}X_1) + bX_1]\} &= \mathcal{E}\exp\{i[(b-a\rho_{21})X_1 + aX_2]\} \\ &= \exp\{-|a|^{\alpha}d_1 - |b|^{\alpha}d_2\}\end{aligned}$$

from which (ii) follows. □

<u>Corollary 3.1</u> *Let* X_1 *and* X_2 *be jointly* SαS *and* $1 < \alpha < 2$. *Then the following are equivalent.*

(*i*) $\mathscr{L}(X_2|X_1)$ *and* $\mathscr{L}(X_1|X_2)$ *are* α*–stable and symmetric.*

(*ii*) X_1 *and* X_2 *are either independent or linearly dependent.*

(*iii*) Γ *is concentrated on* $\{\pm(0,1), \pm(1,0)\}$ *or on* $\pm(c, (1-c^2)^{1/2})$ *for some* $0 \le c \le 1$.

Proof. In view of Proposition 3.1, (i) implies that Γ is concentrated on the set $\{\pm(0,1), \pm((1+\rho_{21}^2)^{-1/2}, \rho_{21}(1+\rho_{21}^2)^{-1/2})\}$ and also on the set $\{\pm(1,0), \pm(\rho_{12}(1+\rho_{12}^2)^{-1/2}, (1+\rho_{12}^2)^{-1/2})\}$, from which (iii) follows. The converse is clear, as is the equivalence of (ii) and (iii). □

Proof of Theorem 3.1. Because of symmetry we will consider $\mathscr{M}_\alpha^{(\ell)}$ only.

(i) Since for all $s < t$ in T, $\mathscr{L}\{X(t)|X(s)\}$ is α–stable and symmetric, it follows from Proposition 3.1 (cf. (3.5)) that

$$\mathscr{E}\{\exp[irX(t)]|X(s)\} = \exp\{-|r|^\alpha\| X(t)-\rho_{ts}X(s)\|_\alpha^\alpha + ir\rho_{ts}X(s)\} \tag{3.6}$$

where $\rho_{ts} = R(t,s)/R(s,s)$ and $\|X(t)-\rho_{ts}X(s)\|_\alpha^\alpha = R(t,t) - |R(t,s)|^\alpha/R(s,s)^{\alpha-1}$ (cf. (3.4)). Since X is Markov and $R(t,s) \ne 0$ for $s < t$, by Theorem 2.1, R(t,s) has the representation (2.5), which implies

$$\rho_{t,s} = \frac{H(t)}{H(s)},$$

$$\|X(t)-\rho_{ts}X(s)\|_\alpha^\alpha = H(t)K(t)^{<\alpha-1>} - |H(t)|^\alpha\{\frac{K(s)}{H(s)}\}^{\alpha-1} = |H(t)|^\alpha\{\tau(t)-\tau(s)\},$$

where $\tau(t) = \{K(t)/H(t)\}^{\alpha-1}$. Thus

$$\mathscr{E}\{\exp[irX(t)]|X(s)\} = \exp\{-|r|^\alpha|H(t)|^\alpha[\tau(t)-\tau(s)] + ir\frac{H(t)}{H(s)}X(s)\},$$

which is the same as the conditional characteristic function for $Y(t) = H(t)L(\tau(t))$, $t \in T$, given in (2.13). It follows that $\mathscr{L}\{X(t)|X(s)\} = \mathscr{L}\{Y(t)|Y(s)\}$ for all $s < t$. It is also clear from (2.8) that $\|X(t)\|_\alpha^\alpha = \|Y(t)\|_\alpha^\alpha$, so that $\mathscr{L}\{X(t)\} = \mathscr{L}\{Y(t)\}$ for all t, and since both X and Y are Markov, $\mathscr{L}(X) = \mathscr{L}(Y)$. The function H may be taken positive without loss of generality.

(ii) If X is stationary, by Corollary 2.1, $R(t,s) = R(0)e^{-\lambda(t-s)}$ for all $s < t$, for some $0 \le \lambda \le \infty$. When $\lambda = +\infty$, $R(t,s) = 0$ for all $s < t$, and by Corollary 3.1 it follows that $\mathscr{L}(X) = \mathscr{L}(I_a)$ for some $a > 0$. When $0 \le \lambda < \infty$, part (i) of the theorem is applicable, necessarily with $H(t) = ae^{-\lambda t}$ and $\tau(t) = be^{\alpha\lambda t}$ for some $a,b > 0$. This completes the proof. □

It is worth mentioning that in the Gaussian case (α=2), clearly $\mathscr{M}_2^{(\ell)} = \mathscr{M}_2^{(r)}$. In contrast when $1 < \alpha < 2$ the common processes of $\mathscr{M}_\alpha^{(\ell)}$ and $\mathscr{M}_\alpha^{(r)}$ are few and trivial.

Theorem 3.2 (*i*) *There is a one–to–one correspondence between* $\mathscr{M}_\alpha^{(\ell)}$ and $\mathscr{M}_\alpha^{(r)}$ *given by* $X(t) = Y(\tau(t))$, $t \in T$, $X \in \mathscr{M}_\alpha^{(\ell)}$, $Y \in \mathscr{M}_\alpha^{(r)}$, *for any fixed function* τ: $T \to T$ *one–to–one, onto, and such that* $\tau(s) > \tau(t)$ *if* $s < t$ (e.g. $\tau(t) = -t$ *when* $T = \mathbb{R}^1$).

(*ii*) *A process* X *belongs to* $\mathscr{M}_\alpha^{(\ell)} \cap \mathscr{M}_\alpha^{(r)}$, $1 < \alpha < 2$, *if and only if it is of the following form: for some finite or denumerable set of intervals* $\{I_n\}_{n=1}^N$ *in* T, $X(t) = a(t)X_n$ *a.s. for each* $t \in I_n$, $n=1,\dots,N$, *where* a *is a real function, nonvanishing in the interior of* I_n *and the r.v.'s* $\{(X_n)_{n=1}^N, X(t),\ t \in T \setminus \cup_{n=1}^N I_n\}$ *are independent.*

(*iii*) *The only stationary processes in* $\mathscr{M}_\alpha^{(\ell)} \cap \mathscr{M}_\alpha^{(r)}$, $1 < \alpha < 2$, *are the processes* I_a *with iid r.v.'s and the constant processes* ($X(t) = aZ$, $-\infty < t < \infty$, $a > 0$, Z: SαS *r.v.*).

Proof. (i) is clear.

(ii) Suppose $X \in \mathscr{M}_\alpha^{(\ell)} \cap \mathscr{M}_\alpha^{(r)}$. Then by Corollary 3.1, for each s,t in T, the r.v.'s X(s) and X(t) are either independent or linearly dependent. But since X is Markov, if X(t) is a multiple of X(s), it will necessarily be a multiple of each X(u) for u in between s and t. Hence for each $t \in T$ there is an interval I_t such that for all $s \in I_t$, X(s) is a multiple of

$X(t)$. Denoting by $\{I_n\}_{n=1}^N$ those intervals among the I_t, $t \in T$, with positive Lebesgue measure we obtain the result.

(iii) In view of stationarity, if two r.v.'s of the process are linearly dependent, they will all be a.s. equal; and if two r.v.'s of the process are independent, they will all be independent. $\square$

4. SUB–GAUSSIAN PROCESSES

Another important class of SαS processes is the class of sub–Gaussian processes which is defined by

$$X(t) = A^{1/2}G(t), \quad t \in T, \tag{4.1}$$

where $G = \{G(t),\ t \in T\}$ is any Gaussian process and A is a totally skewed to the right $(\alpha/2)$-stable random variable (see Feller (1968)) which is positive with probability 1 and satisfies

$$\mathcal{E} \exp\{-uA\} = \exp\{-u^{\alpha/2}\}, \quad u > 0. \tag{4.2}$$

A sub–Gaussian process (4.1) is easily seen to be a SαS process, and when $T = \mathbb{R}^1$, $X(t)$ is stationary if and only if $G(t)$ is.

Our task in this section is to investigate if we can find a Markov, or at least weakly Markov process among sub–Gaussian processes. The obvious candidates are the sub–Gaussian processes given by (4.1) with G being Gaussian Markov process. For example, $G(t)$ could be a Brownian motion; in this case we call the corresponding process $X(t)$ a *sub–Brownian motion*. If $T = \mathbb{R}^1$ and $G(t)$ is Ornstein–Uhlenbeck process, we call $X(t)$ *sub–Ornstein–Uhlenbeck process*. The first result characterizes the weakly Markov sub–Gaussian processes.

<u>Theorem 4.1</u>. *Let X be a sub–Gaussian* SαS *process given by* (4.1). *Then the following are equivalent.*

(*i*) X *is left weakly Markov.*

(*ii*) X *is right weakly Markov.*

(*iii*) G *is Markov.*

Proof. We note for future reference the following fact (see [5]): If (Y_1, Y_2) is a zero-mean Gaussian vector in $\mathbb{R}^2$ with $\mathcal{E} Y_1^2 = \mathcal{E} Y_2^2 = 1$, $\mathcal{E} Y_1 Y_2 = \rho$, and if $X_1 = A^{1/2} Y_1$, $X_2 = A^{1/2} Y_2$, where A is distributed according to (4.2) and is independent of (Y_1, Y_2), then

$$\text{(4.3)} \qquad \mathrm{Cov}[X_1, X_2] = 2^{-\alpha/2} \rho.$$

Assume (i). Then by Theorem 2.1 the covariation function of X satisfies (2.4) and by(4.3) we conclude that the covariance function of the Gaussian process G in (4.1) satisfies the same relation. Thus G must be Markov (since it is Gaussian), i.e. (iii) holds.

Conversely, assume (iii); we will show (i). For any $s < t$, any $u_1, u_2, \ldots, u_k \leq s$ and any $a_1, a_2, \ldots, a_k$, we have by (4.3),

$$\mathrm{Cov}[X(t) - \frac{R(t,s)}{R(s,s)} X(s),\ \Sigma_{i=1}^k a_i X(u_i)]$$

$$= \mathrm{Cov}[A^{1/2}\{G(t) - \frac{\mathcal{E}(G(t)G(s))}{\mathcal{E}(G^2(s))} G(s)\},\ A^{1/2} \Sigma_{i=1}^k a_i G(u_i)]$$

$$= 2^{-\alpha/2} [\mathcal{E}\{\Sigma_{i=1}^k a_i G(u_i)\}^2]^{(\alpha-2)/2} \mathrm{Cov}[G(t) - \frac{\mathcal{E}(G(t)G(s))}{\mathcal{E}(G^2(s))} G(s),\ \Sigma_{i=1}^k a_i G(u_i)] = 0$$

since G is Markov. Therefore, by Theorem 2.1, X is left weakly Markov, i.e. (i) holds. The equivalence of (ii) and (iii) is now deduced by the symmetry. □

Corollary 4.1. *Any sub-Gaussian weakly Markov* $S\alpha S$ *process with covariation function* $R(t,s) \neq 0$ *for any* t,s, *is equivalent to a nondecreasing time change of sub-Brownian motion.*

Proof. Formula (4.3) implies that the covariation function of a sub-Gaussian process determines its finite dimensional distributions. Therefore, we have only to demonstrate that the covariation function of a sub-Gaussian weakly Markov SαS process can be realized by a time changed sub-Brownian motion. This is easy. By Theorem 2.1 and (4.3), the covariation function of a sub-Gaussian weakly Markov SαS process can be represented as $R(t,s) = H(t)K(s)^{<\alpha-1>}$ for all s,t, where $K(t)/H(t)$ is positive and nondecreasing on T. Let $G_1(t) = H(t)B(\tau(t))$, where B is the standard Brownian motion, and $\tau(t) = 2[K(t)/H(t)]^{2(\alpha-1)/\alpha}$. It is straightforward to check using (4.3) that the process $X_1(t) = A^{1/2}G_1(t)$ has $R(t,s)$ as its covariation function. Since X_1 is a time change of sub-Brownian motion, the proof is complete. □

Among all SαS sub-Gaussian processes only very few and highly degenerate are Markov. All these processes are characterized in the following simple fashion. We say that a deterministic function $a(t)$, $t \in T$, *is born at the level* a_1 *and dies at the level* a_2 if for some $s_1 < s_2$, which may be boundary points of T, $a(t) = a_1$ for $t < s_1$ (or $t \leq s_1$), $a(t) = a_2$ for $t > s_2$ (or $t \geq s_2$), and $a(t) \notin \{a_1, a_2\}$ outside of the above intervals; s_1 is the birth time and s_2 is the death time.

<u>Theorem 4.2</u>. *A SαS sub-Gaussian process given by (4.1) is Markov if and only if the Gaussian process G has one of the following forms:* $G(t) = a(t)Y_1$ *or* $G(t) = a(t)\{Y_1 1(t < t_0) + Y_2 1(t \geq t_0)\}$ *or* $G(t) = a(t)\{Y_1 1(t \leq t_0) + Y_2 1(t > t_0)\}$, $t \in T$, *where* (Y_1, Y_2) *is a jointly Gaussian vector in* $\mathbb{R}^2$, $\mathcal{E} Y_1^2 = \mathcal{E} Y_2^2 = 1$, $t_0 \in T$, *and* $a(t)$ *is a real function that is born and dies at the level zero.*

A technical result precedes the proof of the theorem.

<u>Lemma 4.1</u>. *Let* G_1, G_2, G_3 *be i.i.d. standard normal random variables and let* W *be a positive random variable not equal a.s. to a constant and independent of* G_1, G_2, G_3. *Then there is a Borel set* B *such that* $P(WG_1 \in B) > 0$ *and for any* $x_1 \in B$ *the random variables* WG_2 *and* WG_3 *are not independent given* $WG_1 = x_1$.

Proof. Suppose on the contrary that for almost every value x_1 of WG_1 the r.v.'s WG_1 and WG_2 are independent given $WG_1 = x_1$. Then for almost any x_1, x_2, we have

$$f_1(y|x_1) = f_2(y|x_1,x_2) \tag{4.4}$$

for almost any y, where $f_1(\cdot|x_1)$ is the conditional density of WG_3 given $WG_1 = x_1$, and $f_2(\cdot|x_1,x_2)$ is the conditional density of WG_3 given $WG_1 = x_2$, $WG_2 = x_2$. We have

$$f_1(y|x_1) = \frac{\int_0^\infty (2\pi)^{-1/2} w^{-2} \exp[-(y^2+x_1^2)/(2w^2)]\, dF(w)}{\int_0^\infty w^{-1} \exp[-x_1^2/(2w^2)]\, dF(w)}, \tag{4.5}$$

$$f_2(y|x_1,x_2) = \frac{\int_0^\infty (2\pi)^{-1/2} w^{-3} \exp[-(y^2+x_1^2+x_2^2)/(2w^2)]\, dF(w)}{\int_0^\infty w^{-2} \exp[-(x_1^2+x_2^2)/(2w^2)]\, dF(w)}, \tag{4.6}$$

where F is the distribution function of W. It follows from (4.5) and (4.6) that $f_1(\cdot|\cdot)$ is continuous on $\mathbb{R}^2$, and $f_2(\cdot|\cdot,\cdot)$ is continuous on $\mathbb{R}^3$. We conclude that (4.4) is equivalent to

$$g_3(y^2+x_1^2+x_2^2)\, g_1(x_1^2) = g_2(y^2+x_1^2)\, g_2(x_1^2+x_2^2), \tag{4.7}$$

where

$$g_n(r) = \int_0^\infty w^{-n} \exp[-r/(2w^2)]\, dF(w), \quad r > 0.$$

By Hölder's inequality we conclude that the left hand side of (4.7) is strictly larger than its right hand side for all triples (y,x_1,x_2) of the kind $(0,x_1,0)$. By the continuity we conclude that there is an $\epsilon > 0$ such that (4.7) and, therefore, (4.4), do not hold for the triples $(y,x_1,x_2) \in (-\epsilon,\epsilon)^3$. This contradicts the assumption that for almost any x_1 and x_2, (4.4) is true for almost any y. □

Proof of Theorem 4.2. Suppose that the sub-Gaussian process X given by (4.1) is

Markov. If all r.v.'s of the Gaussian process G are linearly dependent, then $G(t) = a(t)Y_1$ for some standard normal r.v. Y_1. Now assume G has two linearly independent r.v.'s, say $G(t_1)$ and $G(t_2)$, $t_1 < t_2$. Furthermore, assume $\mathcal{E}\{G(t_1)G(t_2)\} \neq 0$, as the argument is even simpler in the case of independence. Then we can write $G(t_2) = a_{21}G_1$, $G(t_1) = a_{11}G_1 + a_{12}G_2$, where G_1, G_2 are i.i.d. standard normal r.v.'s and the coefficients a_{11}, a_{12}, a_{21} are different from zero. Take any $t_3 > t_2$. Then we can write $G(t_3) = a_{31}G_1 + a_{32}G_2 + a_{33}G_3$, where G_3 is a standard normal r.v. independent of (G_1, G_2). The process G is Markov by Theorem 4.1; therefore its covariance function satisfies the relation (2.4). Rewriting this relation for the triple (t_1,t_2,t_3) in terms of our particular representation of $G(t_i)$, i=1,2,3, we obtain

$$(a_{11}a_{31} + a_{12}a_{32})\, a_{21}^2 = (a_{11}a_{21})(a_{21}a_{31}),$$

and since a_{11}, a_{12} and a_{21} are different from zero, we conclude that $a_{32} = 0$.

Recall now that by the assumed Markovianness of X(t), for almost any x_2, $X(t_1)$ and $X(t_3)$ are independent given $X(t_2) = x_2$. Since $a_{32} = 0$, it follows that $a_{12}A^{1/2}G_2$ and $a_{33}A^{1/2}G_3$ are independent given $a_{21}A^{1/2}G_1 = x_1$. Suppose first that $a_{33} \neq 0$. Then it follows that for almost any x_1, $A^{1/2}G_2$ and $A^{1/2}G_3$ are independent given $A^{1/2}G_1 = x_1/a_{21}$. But this is impossible by Lemma 4.1. Therefore $a_{33} = 0$. It follows that $G(t_3) = a_{31}G_1$, and thus for any $t > t_2$, G(t) is linearly dependent on $G(t_2)$. The same argument shows that for any $t < t_1$, G(t) is linearly dependent on $G(t_1)$, and that there is a $t_0 \in [t_1,t_2]$ such that for any $t_1 < t < t_0$, G(t) is linearly dependent on $G(t_1)$, while for any $t_0 < t < t_2$, G(t) is linearly dependent on $G(t_2)$. $G(t_0)$ itself has to be linearly dependent either on $G(t_1)$ or on $G(t_2)$. This proves the dependence structure of G(t) described in the statement of the theorem.

Suppose now that there are points $s_1 < s_2 < s_3$ such that $a(s_1) \neq 0$, $a(s_2) = 0$ and $a(s_3) \neq 0$. Applying the assumed Markovianness of $X(t)$ to the triple (s_1, s_2, s_3) shows that one of the following pairs of r.v.'s is independent: $(A^{1/2}Y_1, A^{1/2}Y_1)$ or $(A^{1/2}Y_2, A^{1/2}Y_2)$ or $(A^{1/2}Y_1, A^{1/2}Y_2)$. The r.v.'s of the first two pairs are obviously dependent, and those of the third pair are also known to be dependent (see Lemma 2.1 in [6]). Therefore $a(t)$ must be born and die at the level zero. Since it is obvious that for any Gaussian process of the above form the corresponding sub-Gaussian process is Markov, the proof of the theorem is complete. □

In the particular case of a stationary sub-Gaussian SαS process we can draw the following simple conclusion from the above results.

Corollary 4.2 *The only weakly Markov stationary sub-Gaussian* SαS *processes are the sub-Ornstein-Uhlenbeck processes and the constant processes. The only Markov stationary sub-Gaussian* SαS *processes are the constant processes.*

Corollary 4.2 shows that the sub-Ornstein-Uhlenbeck process is the only weakly Markov stationary sub-Gaussian process. However, the class of sub-Gaussian processes is not closed under linear combinations of its independent members. Therefore a natural question arises: could we obtain a new weakly Markov stationary SαS process as a linear combination of independent stationary sub-Gaussian processes?

Let $G_1, G_2, \ldots, G_n$ be independent stationary Gaussian processes such that for any t,

$$\mathcal{E}\{G_i(t)G_i(0)\} = \rho_i(t), \quad \rho_i(0) = 1, \quad i=1,2,\ldots,n. \tag{4.8}$$

We assume that all the correlation functions $\rho_1, \rho_2, \ldots, \rho_n$ are different. Let

$A_1, A_2, \ldots, A_n$ be i.i.d. random variables distributed according to (4.2) and independent of the Gaussian processes $G_1, G_2, \ldots, G_n$. Let $b_1, b_2, \ldots, b_n$ be positive real numbers. We will prove that if $n > 1$, *the stationary* $S\alpha S$ *process*

$$(4.9) \qquad X(t) = \Sigma_{i=1}^{n} b_i A_i^{1/2} G_i(t), \quad -\infty < t < \infty,$$

cannot be weakly Markov. We start with the following lemma.

Lemma 4.2. *Let* $\varphi_i : S \to R^1$, $i=1,2,\ldots,n$, *be arbitrary distinct functions on a set* S. *Then there is a finite subset* $\{s_1, s_2, \ldots, s_k\} \subset S$ $(k \le n-1)$ *and real numbers* $\theta_1, \theta_2, \ldots, \theta_k$ *such that*

$$\Sigma_{j=1}^{k} \theta_j \varphi_1(s_j) \neq \Sigma_{j=1}^{k} \theta_j \varphi_i(s_j), \quad \textit{any } i=2,\ldots,n.$$

Proof. We prove the lemma by induction in n. For n=2 the claim of the lemma is trivial, and one can choose k=1 and $\theta_1 = 1$. Suppose that the claim of the lemma is correct for $n = n_0$. We shall prove this claim for n_0+1 functions, $\varphi_1, \varphi_2, \ldots, \varphi_{n_0}$, φ_{n_0+1}. By the assumption of the induction we know that there is a set $\{s_1, s_2, \ldots, s_{k_0}\} \subset S$, and real numbers $\theta_1, \theta_2, \ldots, \theta_{k_0}$, such that

$$\Delta(i) = \Sigma_{j=1}^{k_0} \theta_i \varphi_1(s_j) - \Sigma_{j=1}^{k_0} \theta_j \varphi_i(s_j) \neq 0, \quad \text{any } i=2,\ldots,n_0.$$

If $\Delta(n_0+1) \neq 0$, then there is nothing to prove, so we assume that $\Delta(n_0+1) = 0$. Then, there is an $s_{k_0+1} \in S$ such that $\varphi_1(s_{k_0+1}) \neq \varphi_{n_0+1}(s_{k_0+1})$. Put

$$\gamma_i = \varphi_i(s_{k_0+1}) - \varphi_1(s_{k_0+1}), \quad i=2,\ldots,n_0.$$

Clearly, we can choose a real number θ_{k_0+1} satisfying the following conditions:

(a) $\theta_{k_o+1} \neq 0$,

(b) $\theta_{k_o+1} \neq -\Delta_i/\gamma_i$ for all such $i=2,\ldots,n_o$, for which $\gamma_i \neq 0$.

Then

$$\Sigma_{j=1}^{k_o+1} \theta_j \varphi_1(s_j) \neq \Sigma_{j=1}^{k_o+1} \theta_j \varphi_i(s_j) \quad \text{for any } i=2,\ldots,n_o,\ n_o+1.$$

so that the proof of the lemma is complete. □

Proof of the claim. If the process given by (4.9) were weakly Markov, we would have for any t, any $\tau \geq 0$, any $s_i \geq 0$, $i=1,\ldots,k$, any real numbers $c, \theta_1,\ldots,\theta_k$, that for some $0 \leq \lambda \leq \infty$,

$$\text{(4.10)} \qquad \mathrm{Cov}[X(t+\tau) - e^{-\lambda\tau}X(t),\ X(t) + c\,\Sigma_{j=1}^{k} \theta_j X(t-s_j)] = 0.$$

Since the sub-Gaussian processes $A_i^{1/2}G_i(t)$, $i=1,\ldots,n$, are independent, we conclude by the properties of covariation (see Weron (1984)) and by (4.3) and (4.10) that

$$0 = \Sigma_{i=1}^{n} b_i^{\alpha}\,\mathrm{Cov}[A_i^{1/2}\{G_i(t+\tau) - e^{-\lambda\tau}G_i(t)\},\ A_i^{1/2}\{G_i(t) + c\,\Sigma_{j=1}^{k} \theta_j G_i(t-s_j)\}]$$

$$= 2^{-\alpha/2}\,\Sigma_{i=1}^{n} b_i^{\alpha}\,[\mathrm{Var}\{G_i(t) + c\,\Sigma_{j=1}^{k} \theta_j G_i(t-s_j)\}]^{(\alpha-2)/2} \times$$

$$\text{(4.11)} \qquad \mathrm{Cov}[G_i(t+\tau) - e^{-\lambda\tau}G_i(t),\ G_i(t) + c\,\Sigma_{j=1}^{k} \theta_j G_i(t-s_j)]$$

$$= 2^{-\alpha/2}\,\Sigma_{i=1}^{n} b_i^{\alpha}\,[1 + 2c\,\Sigma_{j=1}^{k} \theta_j \rho_i(s_j) + c^2\,\Sigma_{j=1}^{k}\Sigma_{\ell=1}^{k} \theta_j\theta_\ell\,\rho_i(s_j-s_\ell)]^{(\alpha-2)/2} \times$$

$$[\{\rho_i(\tau)-e^{-\lambda\tau}\} + c\,\Sigma_{j=1}^{k} \theta_j\{\rho_i(s_j+\tau) - e^{\lambda\tau}\,\rho_i(s_j)\}].$$

That is, for any t, any $\tau \geq 0$, any $s_i \geq 0$, $i=1,\ldots,k$, any real $c, \theta_1,\ldots,\theta_k$, we have

$$c\,\Sigma_{i=1}^{n}\,\Sigma_{j=1}^{k}\,\theta_j\{\rho_i(s_j+\tau)-e^{-\lambda\tau}\rho_i(s_j)\}b_i^{\alpha}$$

$$(4.12)\qquad \times\,[1+2c\,\Sigma_{j=1}^{k}\,\theta_j\rho_i(s_j)+c^2\,\Sigma_{j=1}^{k}\,\Sigma_{\ell=1}^{k}\,\theta_j\theta_\ell\rho_i(s_j-s_\ell)]^{(\alpha-2)/2}$$

$$=\Sigma_{i=1}^{n}\,b_i^{\alpha}\,\{e^{-\lambda\tau}-\rho_i(\tau)\}[1+2c\,\Sigma_{j=1}^{k}\,\theta_j\rho_i(s_j)+c^2\,\Sigma_{j=1}^{k}\,\Sigma_{\ell=1}^{k}\,\theta_j\theta_\ell\rho_i(s_j-s_\ell)]^{(\alpha-2)/2}.$$

Since the correlation functions $\rho_1,\dots,\rho_n$ are all different, we can find a τ such that not all the expressions $e^{-\lambda\tau}-\rho_i(\tau)$, $i=1,\dots,n$, are equal to zero. We can assume, without loss of generality, that $e^{-\lambda\tau}-\rho_i(\tau)\neq 0$ for $i=1,\dots,n_0$, and $e^{-\lambda\tau}-\rho_i(\tau)=0$ for $i=n_0+1,\dots,n$, for some $1\le n_0\le n$. According to Lemma 4.2, there are points $s_i\ge 0$, and real numbers θ_i, $i=1,\dots,k$, such that

$$\Sigma_{j=1}^{k}\,\theta_j\rho_1(s_j)\neq\Sigma_{j=1}^{k}\,\theta_j\rho_i(s_j),\qquad i=1,\dots,n.$$

Then for these *fixed* values of $\tau,s_1,\dots,s_k,\theta_1,\dots,\theta_k$, we can rewrite (4.12) as

$$(4.13)\qquad c\,\Sigma_{i=1}^{n}\,a_i^{(1)}(1+2c\mu_i+c^2\sigma_i^2)^{(\alpha-2)/2}=\Sigma_{i=1}^{n_0}\,a_i^{(2)}(1+2c\mu_i+c^2\sigma_i^2)^{(\alpha-2)/2}$$

for certain real numbers $a_i^{(1)}$, μ_i, σ_i, $i=1,\dots,n$, $a_i^{(2)}$, $i=1,\dots,n_0$. Then the fact that for any $0<d<1/2$, any p, the following family of functions of c,

$$\{\,(1+2c\mu_i+c^2\sigma_i^2)^{-d},\;c(1+2c\mu_i+c^2\sigma_i^2)^{-d},\quad i=1,\dots,p\,\}$$

is linearly independent as long as all the pairs (μ_i,σ_i^2) are different, implies that $a_1^{(2)}=0$. But we know that $a_1^{(2)}\neq 0$. This contradiction proves that no process of the kind (4.9) can be weakly Markov. □

5. MOVING AVERAGES

A SαS moving average is a process of the form

$$X(t) = \int_{-\infty}^{\infty} f(t-s)\, dL(s), \qquad -\infty < t < \infty,$$

where L is an independeeeently scattered SαS measure on $(\mathbb{R}^1, \mathscr{B}^1, \text{Leb})$, i.e. $\{L(s), -\infty < s < \infty\}$ is a SαS Lévy motion, and $f \in L_\alpha(\mathbb{R}^1, \mathscr{B}^1, \text{Leb})$. $X(t)$ is clearly a stationary process. When f vanishes on the negative line, X is a nonanticipating moving average and

$$X(t) = \int_{-\infty}^{t} f(t-s)\, dL(s). \tag{5.1}$$

A nonanticipating moving average is called invertible if $\overline{\text{sp}}\{X(t),\ t \le \tau\} = \overline{\text{sp}}\{\Delta L(t),\ t \le \tau\} \overset{\Delta}{=} \overline{\text{sp}}\{L(t)-L(s),\ s<t\le\tau\}$ for all τ, i.e. the increments of L represent the innovations of X.

We now determine all nonanticipating, invertible SαS moving averages which are left weakly Markov; these are in fact Markov, and we shall show that they are precisely the SαS Ornstein–Uhlenbeck processes, thus obtaining an integral representation for these processes.

Theorem 5.1 X *is a left weakly Markov, nonanticipating, invertible,* SαS *moving average with* $1 < \alpha < 2$, *if and only if it is of the form*

$$X(t) = a \int_{-\infty}^{t} e^{-\lambda(t-s)} dL(s), \quad -\infty < t < \infty, \tag{5.2}$$

for some $0 < a,\ \lambda < \infty$, *if and only if it is a* SαS *Ornstein–Uhlenbeck process.*

Proof. Let X be a nonanticipating, invertible SαS moving average as given by (5.1). Then X is a left weakly Markov iff for some $0 \le \lambda \le \infty$,

$$\text{Cov}[X(\tau) - e^{-\lambda\tau}X(0),\ Y] = 0$$

for all $\tau>0$ and all $Y \in \overline{sp}\{X(t),\ t\leq 0\} = \overline{sp}\{\Delta L(t),\ t\leq 0\}$, i.e. for all $Y = \int_{-\infty}^{0} g\, dL$ where $\int_{-\infty}^{0} |g(s)|^{\alpha} ds < \infty$; i.e. iff $\forall\, \tau>0$, $\forall\, g \in L_{\alpha}((-\infty,0), \mathscr{B}(-\infty,0), \mathrm{Leb})$,

$$0 = \mathrm{Cov}\left[\int_{-\infty}^{\tau} f(\tau-s)dL(s) - e^{-\lambda\tau}\int_{-\infty}^{0} f(-s)dL(s),\ \int_{-\infty}^{0} g(s)dL(s)\right]$$

$$= \int_{-\infty}^{0} f(\tau-s)g(s)^{\langle\alpha-1\rangle}ds - e^{-\lambda\tau}\int_{-\infty}^{0} f(-s)g(s)^{\langle\alpha-1\rangle}ds.$$

Now putting $g(s) = 1_{(-x,0)}(s)$, $x>0$, we obtain that if X is left weakly Markov then for all $\tau>0$, $x>0$,

$$\int_{-x}^{0} f(\tau-s)ds = e^{-\lambda\tau}\int_{-x}^{0} f(-s)ds,$$

i.e.

$$\int_{\tau}^{\tau+x} f(u)du = e^{-\lambda\tau}\int_{0}^{x} f(u)du,$$

and putting $F(x) = \int_0^x f(u)du$,

$$F(\tau+x) = F(\tau) + e^{-\lambda\tau}F(x), \quad \forall\, \tau, x>0.$$

The parameter λ cannot take the values 0 or $+\infty$. Indeed, if $\lambda = +\infty$ then $F(\tau+x) = F(\tau)$, $\forall\, \tau, x>0$, implies $F(x) = \mathrm{Const.}$, $x>0$, hence $f(x)=0$ a.e. on $(0,\infty)$ and $X(t)=0$ a.s. for all t, i.e. the process X is identically zero. On the other hand, if $\lambda=0$ then $F(\tau+x) = F(\tau)+F(x)$, $\forall\, \tau, x>0$, implies $F(x) = cx$, $x\geq 0$, hence $f(x)=c$ a.s. on $(0,\infty)$ which does not belong to $L_{\alpha}(\mathrm{Leb})$ unless $c=0$ in which case again the process X is identically zero. It follows that $0<\lambda<\infty$. Interchanging τ and x we also have

$$F(\tau+x) = F(x) + e^{-\lambda x}F(\tau), \quad \forall\, \tau,\ x>0,$$

and thus

$$\frac{F(x)}{1-e^{-\lambda x}} = \frac{F(\tau)}{1-e^{-\lambda\tau}}, \quad \forall\, \tau, x>0.$$

Hence $F(x) = c(1-e^{\lambda x})$, $x > 0$, and $f(x) = c\lambda e^{-\lambda x}$ a.e. on $(0,\infty)$. In view of the symmetry of the distribution of X, the finite constant $a = c\lambda$ may be taken positive. Thus (5.2) is shown.

Conversely, it is clear that the process (5.2) satisfies the necessary and sufficient condition for being left weakly Markov; is invertible, in fact it is the stationary solution of the stochastic differential equation $X'(t) + \lambda X(t) = aL'(t)$, where $L'(t)$ is SαS white noise; and is Markov, as follows from $(s < t)$

$$X(t) = e^{-\lambda(t-s)}X(s) + a\int_s^t e^{-\lambda(t-u)}dL(u) \tag{5.3}$$

where the second term is independent of $\{X(u), u \le s\}$.

We finally show that the process (5.2) is the SαS Ornstein-Uhlenbeck process

$$Y(t) = \frac{ae^{-\lambda t}}{(\alpha\lambda)^{1/\alpha}} L(e^{\alpha\lambda t}), \quad -\infty < t < \infty .$$

Indeed

$$\|X(t)\|_\alpha^\alpha = a^\alpha \int_{-\infty}^t e^{-\alpha\lambda(t-s)}ds = \frac{a^\alpha}{\alpha\lambda},$$

$$\|Y(t)\|_\alpha^\alpha = \frac{a^\alpha e^{-\alpha\lambda t}}{\alpha\lambda} e^{\alpha\lambda t} = \frac{a^\alpha}{\alpha\lambda},$$

and thus $\mathscr{L}\{X(t)\} = \mathscr{L}\{Y(t)\}$. From (5.3), using the independence of the two terms on the right hand side we have for $s < t$,

$$\mathscr{E}\{\exp[irX(t)]\,|X(s)\} = \exp\{ire^{-\lambda(t-s)}X(s) - |ra|^\alpha \int_s^t e^{\alpha\lambda(t-u)}du\}$$

$$= \exp\{ire^{-\lambda(t-s)}X(s) - |r|^\alpha a^\alpha(1-e^{\alpha\lambda(t-s)})/(\alpha\lambda)\}.$$

On the other hand, from (2.13) with $H(t) = ae^{-\lambda t}(\alpha\lambda)^{-1/\alpha}$, $\tau(t) = e^{\alpha\lambda t}$, we have

$$\mathscr{E}\{\exp[irY(t)]\,|Y(s)\} = \exp\{ire^{-\lambda(t-s)}Y(s) - |r|^\alpha a^\alpha(1-e^{-\alpha\lambda(t-s)})/(\alpha\lambda)\}.$$

Therefore $\mathscr{L}\{X(t)|X(s)\} = \mathscr{L}\{Y(t)|Y(s)\}$ for all $s < t$. Since both X and Y are Markov it follows that $\mathscr{L}(X) = \mathscr{L}(Y)$. It is clear that the S$\alpha$S Ornstein–Uhlenbeck processes Y exhaust the class of all SαS Ornstein–Uhlenbeck processes (cf. (2.14)). □

When X is a SαS moving average and f vanishes on the positive line, we say that X is fully anticipatory and

$$X(t) = \int_t^\infty f(t-s)\, dL(s).$$

A fully anticipatory moving average is called invertible if $\overline{sp}\{X(t),\ t \geq \tau\} = \overline{sp}\{\Delta L(t),\ t \geq \tau\}$ for all τ, i.e. the increments of L represent the backward innovations of X. It is easily seen that X(t) is fully anticipatory and invertible (with kernel $f(\cdot)$) iff X $(-t)$ is nonanticipating and invertible (with kernel $f(-\cdot)$). Thus the only fully anticipatory, invertible SαS moving averages $(1 < \alpha \leq 2)$ which are right weakly Markov are the Markov processes

$$Y(t) = a' \int_t^\infty e^{\lambda'(t-s)} dL(s), \quad -\infty < t < \infty, \tag{5.4}$$

where $0 < a',\ \lambda' < \infty$, which have covariation function

$$R_Y(t) = \begin{cases} R_Y(0)\, e^{-\lambda'(\alpha-1)t}, & t \geq 0, \\ R_Y(0)\, e^{\lambda' t}, & t \leq 0. \end{cases} \tag{5.5}$$

Theorem 3.2 (iii) implies that the two classes of Markov processes introduced in this section, the nonanticipating and the fully anticipatory invertible SαS moving averages, are clearly distinct when $1 < \alpha < 2$.

6. HARMONIZABLE PROCESSES

A complex harmonizable SαS process has a harmonic spectral representation

$$(6.1) \qquad X(t) = \int_{-\infty}^{\infty} e^{itu} dZ(u), \quad -\infty < t < \infty,$$

where Z is a complex, independently scattered, isotropic SαS measure on $(\mathbb{R}^1, \mathscr{B}^1, \mu)$ with μ a finite symmetric measure. For every complex-valued function $g \in L_\alpha(\mu)$ the r.v. $\int g dZ$ is complex isotropic SαS with $\mathscr{E} \exp\{\mathscr{R}e\,\bar{z}\int g dZ\} = \exp\{-|z|^\alpha \int |g|^\alpha d\mu\}$, for complex z. Covariation is given by

$$\mathrm{Cov}[\int g_1 dZ, \int g_2 dZ] = \int g_1 g_2^{<\alpha-1>} d\mu$$

for all $g_1, g_2 \in L_\alpha(\mu)$, where $z^{<q>} = |z|^{q-1}\bar{z}$. All properties of covariation and regression used here in the real case are also valid in this complex isotropic case (see [4]).

The harmonizable process (6.1) is stationary and has covariation function

$$(6.2) \qquad R(t) = \int_{-\infty}^{\infty} e^{itu} d\mu(u).$$

Taking its real part would provide a real harmonizable process, but it is more convenient to work with complex quantities when dealing with Fourier type representations. Note that when $\alpha=2$ all (continuous in probability) stationary Gaussian processes are harmonizable, while when $1 < \alpha < 2$, the harmonizable, the nonanticipating moving averages and the sub-Gaussian SαS processes form disjoint classes [6]. We now show that (nontrivial) harmonizable processes cannot be weakly Markov.

Theorem 6.1. *The only harmonizable* SαS *process with* $1 < \alpha < 2$ *which is left or right weakly Markov is the constant* SαS *process.*

<u>Proof</u>. Let X be harmonizable SαS as in (6.1). Assume furthermore that X is, say, left weakly Markov. Then $R(t) = R(0)e^{-\lambda t}$ for all $t \geq 0$ and some $0 \leq \lambda \leq \infty$. The continuity of R in (6.2) excludes the value $\lambda = \infty$. When $\lambda = 0$, X is a constant SαS process with μ concentrated at 0. Assume from now on that $0 < \lambda < \infty$. Since R is an even function (cf. (6.2)) it follows that

$$R(t) = R(0)\, e^{-\lambda|t|} \quad \text{for all } t,$$

and by (6.2),

$$\frac{d\mu(u)}{du} = \frac{\lambda R(0)}{\pi(\lambda^2+u^2)}\,.$$

Since X is right weakly Markov, it satisfies

$$\mathrm{Cov}[X(\tau) - e^{-\lambda\tau}X(0), Y] = 0, \quad \forall\, \tau \geq 0, \ \forall\, Y \in \overline{\mathrm{sp}}\{X(s), s \leq 0\}.$$

Taking $Y = X(0) + X(-v) = \int_{-\infty}^{\infty} (1+e^{-ivu})dZ(u)$ with $v \geq 0$, we obtain for all τ, $v \geq 0$,

$$0 = \int_{-\infty}^{\infty} (e^{i\tau u} - e^{-\lambda\tau})(1+e^{-ivu})^{<\alpha-1>} d\mu(u)$$

$$= \int_{-\infty}^{\infty} \frac{(e^{i\tau u} - e^{-\lambda\tau})(1+e^{ivu})}{[2(1+\cos vu)]^{1-\alpha/2}} \frac{du}{\lambda^2+u^2}\,.$$

Introduce for each $v \geq 0$ the measure μ_v by

$$\frac{d\mu_v(u)}{du} = \frac{1}{(1+\cos vu)^{1-\alpha/2}\,(\lambda^2+u^2)}\,.$$

Then μ_v is symmetric and finite since for $vu \approx (2k+1)\pi$, $1 + \cos vu \approx [vu-(2k+1)\pi]^2/2$ and $2(1-\alpha/2) = 2 - \alpha < 1$. We have for all τ, $v \geq 0$,

$$\int_{-\infty}^{\infty} e^{i\tau u}(1+e^{ivu})d\mu_v(u) = e^{-\lambda\tau} \int_{-\infty}^{\infty} (1+e^{ivu})d\mu_v(u).$$

Let $f_v(\cdot)$ be the characteristic function of the probability measure $\mu_v/\mu_v(\mathbb{R}^1)$. Since μ_v is symmetric, f_v is real and even, and for all τ, $v \geq 0$,

$$f_v(\tau) + f_v(\tau+v) = e^{-\lambda\tau} \frac{1}{\mu_v(\mathbb{R}^1)} \int_{-\infty}^{\infty} (1+\cos vu)\,d\mu_v(u) \overset{\Delta}{=} e^{-\lambda\tau} b(v).$$

It follows that for all $k=1,2,\ldots,$ and v, $\tau \geq 0$,

$$f_v(\tau+2kv) = f_v(\tau) - e^{\lambda\tau} b(v)\,(1-e^{-\lambda v}+e^{-2\lambda v}-\ldots-e^{-(2k-1)\lambda v})$$

$$= f_v(\tau) - e^{-\lambda\tau} b(v) \frac{1-e^{-2k\lambda v}}{1+e^{-\lambda v}},$$

and thus

$$\lim_{k\to\infty} f_v(\tau+2kv) = f_v(\tau) - e^{-\lambda\tau} \frac{b(v)}{1+e^{-\lambda v}}, \qquad \forall\, \tau,\ v > 0.$$

But since f is the Fourier transform of an absolutely continuous, finite measure, by the Riemann-Lebesgue lemma, $f(\pm\infty) = 0$. Hence $f_v(\tau) = e^{-\lambda\tau} b(v)/(1+e^{-\lambda v})$, $\forall\ \tau,\ v > 0$, and by the symmetry of f_v,

$$f_v(\tau) = \frac{b(v)}{1+e^{-\lambda v}} e^{-\lambda|\tau|}, \quad \forall\ \tau, \quad \forall\ v>0.$$

The uniqueness of Fourier transform now implies that for any $v > 0$,

$$\frac{1}{(1+\cos vu)^{1-\alpha/2}(\lambda^2+u^2)\mu_v(\mathbb{R}^1)} = \frac{b(v)}{1+e^{-\lambda v}} \frac{\lambda}{\pi(\lambda^2+u^2)}, \qquad \text{for almost any } u$$

and thus

$$(1+\cos vu)^{1-\alpha/2} = \frac{\pi(1+e^{-\lambda v})}{\lambda\ b(v)\mu_v(\mathbb{R}^1)}, \quad u \in \mathbb{R}^1,$$

which is a contradiction, as the left hand side depends on u while the right hand side does not. It follows that X with $0 < \lambda < \infty$ cannot be right weakly Markov and similarly it cannot be left weakly Markov. □

The proof of Theorem 4.1 provides an example of a stationary SαS process with covariation function $R(t) = R(0)e^{-\lambda|t|}$ $(0 < \lambda < \infty)$ which is neither right nor left weakly Markov. In fact this is a harmonizable process with representation

$$X(t) = [R(0)\lambda/\pi] \int_{-\infty}^{\infty} e^{itu} \frac{1}{(u^2+\lambda^2)^{1/\alpha}} dZ(u) \tag{6.3}$$

where Z is complex isotropic SαS motion.

7. A FAMILY OF ONE-SIDED WEAKLY MARKOV PROCESSES

Every left weakly Markov process constructed in earlier sections is also right weakly Markov, and vice versa. In this section we construct non-Gaussian SαS processes which are left weakly Markov processes but are not right weakly Markov, and vice versa. Thus for non-Gaussian SαS processes weak Markovianness is not a symmetric property. For simplicity we consider the case of stationary processes.

Lemma 7.1. *Let* X_1 *and* X_2 *be independent left (correspondingly right) weakly Markov stationary* SαS *processes with covariation functions satisfying*

$$R_i(t) = r_i e^{-a|t|}$$

for all $t \geq 0$ *(correspondingly all* $t \leq 0$*),* $i=1,2$*, and some* r_1, r_2, $a > 0$. *Then for any real numbers* b_1, b_2, *the process*

$$X(t) = b_1 X_1(t) + b_2 X_2(t), \quad -\infty < t < \infty,$$

is left (correspondingly right) weakly Markov stationary SαS *process.*

Proof. Only weak Markovianness requires proof. We assume that the original processes are left weakly Markov. For any $\tau \geq 0$, any $u_1, \ldots, u_k \leq 0$, any real $c_1, \ldots, c_k$, we have, using the independence of X_1 and X_2,

$$\begin{aligned}
&\mathrm{Cov}[X(\tau) - e^{-a\tau}X(0), \Sigma_{i=1}^k c_i X(u_i)] \\
&\quad = \mathrm{Cov}[b_1\{X_1(\tau) - e^{-a\tau}X_1(0)\} + b_2\{X_2(\tau) - e^{-a\tau}X_2(0)\}, \\
&\qquad\qquad b_1 \Sigma_{i=1}^k c_i X_1(u_i) + b_2 \Sigma_{i=1}^k c_i X_2(u_i)] \\
&\quad = |b_1|^\alpha \mathrm{Cov}[X_1(\tau) - e^{a\tau}X_1(0), \ \Sigma_{i=1}^k c_i X_1(u_i)] \\
&\quad + |b_2|^\alpha \mathrm{Cov}[X_2(\tau) - e^{-a\tau}X_2(0), \ \Sigma_{i=1}^k c_i X_2(u_i)] = 0
\end{aligned}$$

because of the left weak Markovianness of X_1 and X_2. Theorem 2.1 implies that X is left weakly Markov. □

Recall that we have introduced three families of weakly Markov stationary SαS processes. The SαS Ornstein–Uhlenbeck processes (2.13), the inverted SαS Ornstein–Uhlenbeck processes (3.3), and the sub–Ornstein–Uhlenbeck processes in Section 4.

Fix $a > 0$. Let X_1 be SαS Ornstein–Uhlenbeck process such that

$$R_1(t) = e^{-at}, \quad t \geq 0.$$

Then by (2.15) we have

$$R_1(t) = e^{(\alpha-1)at}, \quad t \leq 0.$$

Similarly, let X_2 be inverted SαS Ornstein–Uhlenbeck process with

$$R_2(t) = e^{-at}, \quad t \geq 0.$$

Then

$$R_2(t) = e^{at/(\alpha-1)}, \quad t \leq 0.$$

Finally, let X_3 be sub–Ornstein–Uhlenbeck process with

$$R_3(t) = e^{-at}, \quad t \geq 0.$$

Then (4.3) implies that

$$R_3(t) = e^{at}, \quad t \leq 0.$$

The three processes X_1, X_2, X_3 are assumed to be independent. Let b_1, b_2, b_3 be nonnegative numbers, at most one of which is equal to zero. Define a new stochastic process

$$X(t) = b_1X_1(t) + b_2X_2(t) + b_3X_3(t), \qquad -\infty < t < \infty . \tag{7.1}$$

By Lemma 7.1, X is left weakly Markov stationary SαS process. Using the independence of X_1, X_2, X_3 we obtain

$$R(t) = \mathrm{Cov}[X(t), X(0)]$$

$$= b_1^\alpha \mathrm{Cov}[X_1(t), X_1(0)] + b_2^\alpha \mathrm{Cov}[X_2(t), X_2(0)] + b_3^\alpha \mathrm{Cov}[X_3(t), X_3(0)]$$

$$= b_1^\alpha R_1(t) + b_2^\alpha R_2(t) + b_3^\alpha R_3(t)$$

$$= \begin{cases} (b_1^\alpha + b_2^\alpha + b_3^\alpha) e^{-at} & , \quad t \geq 0, \\ b_1^\alpha e^{(\alpha-1)at} + b_2^\alpha e^{at/(\alpha-1)} + b_3^\alpha e^{at} & , \quad t \leq 0, \end{cases}$$

and this is not the covariation function of a right weakly Markov stationary SαS process (cf. Section 2). Therefore, (7.1) defines a whole family (with parameters a, b_1, b_2, b_3) of left weakly Markov stationary SαS processes that *are not* right weakly Markov.

It is clear that in a similar way we can define a family of right weakly Markov SαS processes which are not left weakly Markov.

REFERENCES

[1] BORISOV, I. S. (1982) On a criterion for Gaussian random processes to be Markovian. Theor. Probab. Appl. 27, 863–865.

[2] CAMBANIS, S., HARDIN, C. D., JR. and WERON, A. (1987) Ergodic properties of stationary stable processes. Stochastic Proc. Appl. 24, 1–18.

[3] CAMBANIS, S., HARDIN, C. D. JR. and WERON, A. (1985) Innovations and Wold decompositions of stable sequences. Center for Stochastic Processes Tech. Rept. No. 106, Statist. Dept., Univ. of North Carolina.

[4] CAMBANIS, S. and MIAMEE, A. G. (1985) On prediction of harmonizable stable processes. Center for Stochastic Processes Tech. Rept. No. 110, Statistics Dept., Univ of North Carolina.

[5] CAMBANIS, S. and MILLER, G. (1981) Linear problems in p^{th} order and stable processes. SIAM J. Appl. Math. 41, 43–69.

[6] CAMBANIS, S. and SOLTANI, A. R. (1984) Prediction of stable processes: Spectral and moving average representations. Z. Wahrsch. verw. Geb. 66, 593–612.

[7] FELLER, W. (1968) An Introduction to Probabilitty Theory and Its Applications, Vol. I. Wiley, New York.

[8] HARDIN, D. C. JR. (1982) On the spectral representation of symmetric stable processes. J. Multivariate Anal. 12, 385–401.

[9] KANTER, M. (1972) Linear sample spaces and stable processes. J. Funct. Anal. 9, 441–456.

[10] KUELBS, J. (1973) A representation theorem for symmetric stable processes and stable measures on H. Z. Wahrsch. verw. Geb. 26, 259–271.

[11] TIMOSZYK, W. (1974) A characterization of Gaussian processes that are Markovian. Coll. Math. 30, 157–167.

[12] WERON, A. (1984) Stable processes and measures: A survey. Probability Theory on Vector Spaces III. D. Snyzol and A. Weron, Eds., Lecture Notes in Mathematics, Vol. 1080, Springer, 306–364.

[13] WONG, E. and HAJEK, B. (1985) Stochastic Processes in Engineering Systems. Springer, New York.

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